ON UNIVERSAL MINIMAL COMPACT G-SPACES

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Abstract. For every topological group G one can define the universal minimal compact G-space $X = M_G$ characterized by the following properties: (1) X has no proper closed G-invariant subsets; (2) for every compact G-space Y there exists a G-map $X \to Y$. If G is the group of all orientation-preserving homeomorphisms of the circle S^1, then M_G can be identified with S^1 (V. Pestov). We show that the circle cannot be replaced by the Hilbert cube or a compact manifold of dimension > 1. This answers a question of V. Pestov. Moreover, we prove that for every topological group G the action of G on M_G is not 3-transitive.

1. Introduction

With every topological group G one can associate the *universal minimal compact G-space* M_G. To define this object, recall some basic definitions. A *G-space* is a topological space X with a continuous action of G, that is, a map $G \times X \to X$ satisfying $g(hx) = (gh)x$ and $1x = x$ ($g, h \in G$, $x \in X$). A G-space X is *minimal* if it has no proper G-invariant closed subsets or, equivalently, if the orbit Gx is dense in X for every $x \in X$. A map $f : X \to Y$ between two G-spaces is *G-equivariant*, or a *G-map* for short, if $f(gx) = gf(x)$ for every $g \in G$ and $x \in X$.

All *maps* are assumed to be continuous, and 'compact' includes 'Hausdorff'. The universal minimal compact G-space M_G is characterized by the following property: M_G is a minimal compact G-space, and for every compact minimal G-space X there exists a G-map of M_G onto X. Since Zorn's lemma implies that every compact G-space has a minimal compact G-subspace, it follows that for every compact G-space X, minimal or not, there exist a G-map of M_G to X.

The existence of M_G is easy: consider the product of a representative family of compact minimal G-spaces, and take any minimal closed G-subspace of this product for M_G. It is also true that M_G is unique, in the sense that any two universal minimal compact G-spaces are isomorphic [1]. For the reader's convenience, we give a proof of this fact in the Appendix.

If G is locally compact, the action of G on M_G is free [7] (see also [5], Theorem 3.1.1), that is, if $g \neq 1$, then $gx \neq x$ for every $x \in M_G$. On the other hand, M_G is a singleton for many naturally arising non-locally compact groups G. This property of G is equivalent to the following *fixed point on compacta (f.p.c.) property*: every compact G-space has a G-fixed point. (A point x is G-fixed if $gx = x$ for all $g \in G$.) For example, if H is a Hilbert space, the group $U(H)$ of all unitary operators on H, equipped with the pointwise convergence topology, has the f.p.c. property (Gromov

Date: June 10, 2000.

2000 Mathematical Subject Classification. Primary 22F05. Secondary 22A05, 22A15, 54D30, 54H15, 57S05, 57S25.

– Milman); another example of a group with this property, due to Pestov, is $H_+(\mathbb{R})$, the group of all orientation-preserving self-homeomorphisms of the real line. We refer the reader to beautiful papers by V. Pestov [3, 4, 5] on this subject.

Let S^1 be a circle, and let $G = H_+(S^1)$ be the group of all orientation-preserving self-homeomorphisms of S^1. Then M_G can be identified with S^1 [3], Theorem 6.6. The question arises whether a similar assertion holds for the Hilbert cube Q. This question is due to V. Pestov, who writes in [3], Concluding Remarks, that his theorem "tends to suggest that the Hilbert cube I^ω might serve as the universal minimal flow for the group Homeo (I^ω)". In other words, let $G = H(Q)$ be the group of all self-homeomorphisms of $Q = I^\omega$, equipped with the compact-open topology. Are M_G and Q isomorphic as G-spaces?

The aim of the present paper is to answer this question in the negative. Let us say that the action of a group G on a G-space X is *3-transitive* if $|X| \geq 3$ and for any triples (a_1, a_2, a_3) and (b_1, b_2, b_3) of distinct points in X there exists $g \in G$ such that $ga_i = b_i$, $i = 1, 2, 3$.

Theorem 1.1. *For every topological group G the action of G on the universal minimal compact G-space M_G is not 3-transitive.*

Since the action of $H(Q)$ on Q is 3-transitive, it follows that $M_G \neq Q$ for $G = H(Q)$. Similarly, if K is compact and G is a 3-transitive group of homeomorphisms of K, then $M_G \neq K$. This remark applies, for example, if K is a manifold of dimension > 1 or a Menger manifold and $G = H(K)$.

Question 1.2. Let $G = H(Q)$. Is M_G metrizable?

A similar question can be asked when Q is replaced by a compact manifold or a Menger manifold.

Let P be the pseudoarc (= the unique hereditarily indecomposable chainable continuum) and $G = H(P)$. The action of G on P is transitive but not 2-transitive, and the following question remains open:

Question 1.3. Let P be the pseudoarc and $G = H(P)$. Can M_G be identified with P?

2. Proof of the main theorem

The proof of Theorem 1.1 depends on the consideration of the space of maximal chains of closed sets. For a compact space K let $\operatorname{Exp} K$ be the (compact) space of all non-empty closed subsets of K, equipped with the Vietoris topology. A subset $C \subset \operatorname{Exp} K$ is a *chain* if for any $E, F \in C$ either $E \subset F$ or $F \subset E$. If $C \subset \operatorname{Exp} K$ is a chain, so is the closure of C. It follows that every maximal chain is a closed subset of $\operatorname{Exp} K$ and hence an element of $\operatorname{Exp}\operatorname{Exp} K$. Let $\Phi \subset \operatorname{Exp}\operatorname{Exp} K$ be the space of all maximal chains. Then Φ is closed in $\operatorname{Exp}\operatorname{Exp} K$ and hence compact. Let us sketch a proof. Clearly the closure of Φ consists of chains. Assume $C \in \operatorname{Exp}\operatorname{Exp} K$ is a non-maximal chain. We construct a neighbourhood $\mathfrak{W}$ of C in $\operatorname{Exp}\operatorname{Exp} K$ which is disjoint from Φ. One the following cases holds: (1) the first member of C has more than one point, or (2) the last member of C is not K, or (3) the chain C contains "big gaps": there are $F_1, F_2 \in C$ such that $|F_2 \setminus F_1| \geq 2$ and for every

$F \in C$ either $F \subset F_1$ or $F_2 \subset F$. For example, consider the third case (the first two cases are simpler). Find open sets U, V_1, V_2 in K with pairwise disjoint closures such that $F_1 \subset U$ and F_2 meets both V_1 and V_2. Let $\mathfrak{W} = \{D \in \operatorname{Exp}\operatorname{Exp} K :$ every member of D either is contained in U or meets both V_1 and $V_2\}$. Then $\mathfrak{W}$ is a neighbourhood of C which does not meet Φ. Indeed, suppose $D \in \mathfrak{W} \cap \Phi$. Let E_1 be the largest member of D which is contained in $\bar{U}$. Let E_2 be the smallest member of D which meets both $\bar{V}_1$ and $\bar{V}_2$. For every $E \in D$ we have either $E \subset E_1$ or $E_2 \subset E$, and $|E_2 \setminus E_1| \geq 2$. Pick a point $p \in E_2 \setminus E_1$. The set $E_1 \cup \{p\}$ is comparable with every member of D but is not a member of D. This contradicts the maximality of D. We have proved that Φ is compact.

Suppose G is a topological group and K is a compact G-space. Then the natural action of G on $\operatorname{Exp} K$ is continuous, hence $\operatorname{Exp} K$ is a compact G-space, and so is $\operatorname{Exp}\operatorname{Exp} K$. Since the closed set $\Phi \subset \operatorname{Exp}\operatorname{Exp} K$ is G-invariant, Φ is a compact G-space, too.

Proposition 2.1. *Let G be a topological group. Pick $p \in M_G$, and let $H = \{g \in G : gp = p\}$ be the stabilizer of p. There exists a maximal chain C of closed subsets of M_G such that C is H-invariant: if $F \in C$ and $g \in H$, then $gF \in C$.*

Note that members of an H-invariant chain need not be H-invariant.

Proof. Every compact G-space X has an H-invariant point. Indeed, there exists a G-map $f : M_G \to X$, and since p is H-invariant, so is $f(p) \in X$.

Let $\Phi \subset \operatorname{Exp}\operatorname{Exp} M_G$ be the compact space of all maximal chains of closed subsets of M_G. We saw that Φ is a compact G-space. Thus Φ has an H-invariant point. $\square$

Theorem 1.1 follows from Proposition 2.1:

Proof of Theorem 1.1. Assume that the action of G on $X = M_G$ is 3-transitive. Pick $p \in X$, and let $H = \{g \in G : gp = p\}$. According to Proposition 2.1, there exists an H-invariant maximal chain C of closed subsets of X. The smallest member of C is an H-invariant singleton. Since G is 2-transitive on X, the only H-invariant singleton is $\{p\}$. Thus $\{p\} \in C$, and all members of C contain p. Our definition of 3-transitivity implies that $|X| \geq 3$. Thus there exists $F \in C$ such that $F \neq \{p\}$ and $F \neq X$. Pick $a \in F \setminus \{p\}$ and $b \in X \setminus F$. The points p, a, b are all distinct. Since G is 3-transitive on X, there exists $g \in G$ such that $gp = p$, $ga = b$ and $gb = a$. Since $a \in F$ and $b \notin F$, we have $b = ga \in gF$ and $a = gb \notin gF$. Thus $a \in F \setminus gF$ and $b \in gF \setminus F$, so F and gF are not comparable. On the other hand, the equality $gp = p$ means that $g \in H$. Since C is H-invariant and $F \in C$, we have $gF \in C$. Hence F and gF must be comparable, being members of the chain C. We have arrived at a contradiction. $\square$

Example 2.2. Consider the group $G = H_+(S^1)$ of all orientation-preserving self-homeomorphisms of the circle S^1. According to Pestov's result cited above, $M_G = S^1$. This example shows that the action of G on M_G can be 2-transitive. Pick $p \in S^1$, and let $H \subset G$ be the stabilizer of p. Proposition 2.1 implies that there must exist H-invariant maximal chains of closed subsets of S^1. It is easy to see that there are precisely two such chains. They consist of the singleton $\{p\}$, the whole circle and of all arcs that either "start at p" or "end at p", respectively.

Remark 2.3. Let P be the pseudoarc, and let $G = H(P)$. Pick a point $p \in P$, and let $H \subset G$ be the stabilizer of p. Then there exists an H-invariant maximal chain C of closed subsets of P. Namely, let C be the collection of all subcontinua $F \subset P$ such that $p \in F$. Since any two subcontinua of P are either disjoint or comparable, it follows that C is a chain. The chain C can be shown to be maximal, and it is obvious that C is H-invariant. Thus Proposition 2.1 does not contradict the conjecture that $M_G = P$. This observation motivates our question 1.3.

3. Appendix: Uniqueness of M_G

We sketch a proof of the uniqueness of M_G up to a G-isomorphism.

Let G be a topological group. The *greatest ambit* $X = \mathcal{S}(G)$ for G is a compact G-space with a distinguished point e such that for every pointed compact G-space (Y, e') there exists a unique G-morphism $f : X \to Y$ such that $f(e) = e'$. The greatest ambit is defined uniquely up to a G-isomorphism preserving distinguished points. We can take for $\mathcal{S}(G)$ the Samuel compactification of G equipped with the right uniformity, which is the compactification of G corresponding to the algebra of all bounded right uniformly continuous functions. The distinguished point is the unity of G. See [3, 4, 5] for more details.

The greatest ambit X has a natural structure of a left-topological semigroup. This means that there is an associative multiplication $(x, y) \mapsto xy$ on X (extending the original multiplication on G) such that for every $y \in X$ the self-map $x \mapsto xy$ of X is continuous. Let $x, y \in X$. There is a unique G-map $r_y : X \to X$ such that $r_y(e) = y$. Define $xy = r_y(x)$. If y is fixed, the map $x \mapsto xy$ is equal to r_y and hence is continuous. If $y, z \in X$, the self-maps $r_z r_y$ and r_{yz} of X are equal, since both are G-maps sending e to $yz = r_z(y)$. This means that the multiplication on X is associative. The distinguished element $e \in X$ is the unity of X: we have $ex = r_x(e) = x$ and $xe = r_e(x) = x$. If $g \in G$ and $x \in X$, the expression gx can be understood in two ways: in the sense of the exterior action of G on X and as a product in X; these two meanings agree. If $f : X \to X$ is a G-self-map and $a = f(e)$, then $f(x) = f(xe) = xf(e) = xa = r_a(x)$ for all $x \in X$. Thus the semigroup of all G-self-maps of X coincides with the semigroup $\{r_y : y \in X\}$ of all right multiplications.

A subset $I \subset X$ is a *left ideal* if $XI \subset I$. Closed G-subspaces of X are the same as closed left ideals of X. An element x of a semigroup is an *idempotent* if $x^2 = x$. Every closed G-subspace of X, being a left ideal, is moreover a left-topological compact semigroup and hence contains an idempotent, according to the following fundamental result of R. Ellis (see [6], Proposition 2.1 or [2], Theorem 3.11):

Theorem 3.1. *Every non-empty compact left-topological semigroup K contains an idempotent.*

Proof. Zorn's lemma implies that there exists a minimal element Y in the set of all closed non-empty subsemigroups of K. Fix $a \in Y$. We claim that $a^2 = a$ (and hence Y is a singleton). The set Ya, being a closed subsemigroup of Y, is equal to Y. It follows that the closed subsemigroup $Z = \{x \in Y : xa = a\}$ is non-empty. Hence $Z = Y$ and $xa = a$ for every $x \in Y$. In particular, $a^2 = a$. □

Let M be a minimal closed left ideal of X. We have just proved that there is an idempotent $p \in M$. Since Xp is a closed left ideal contained in M, we have $Xp = M$. Thus the G-map $r_p : X \to M$ defined by $r_p(x) = xp$ is a retraction of X onto M. In particular, $xp = x$ for every $x \in M$.

Proposition 3.2. *Every G-map $f : M \to M$ has the form $f(x) = xy$ for some $y \in M$*

Proof. The composition $h = fr_p : X \to M$ is a G-map of X into itself, hence it has the form $h = r_y$, where $y = h(e) \in M$. Since $r_p \restriction M = \mathrm{Id}$, we have $f = h \restriction M = r_y \restriction M$. □

Proposition 3.3. *Every G-map $f : M \to M$ is bijective.*

Proof. According to Proposition 3.2, there is $a \in M$ such that $f(x) = xa$ for all $x \in M$. Since Ma is a compact G-space contained in M, we have $Ma = M$ by the minimality of M. Thus there exists $b \in M$ such that $ba = p$. Let $g : M \to M$ be the G-map defined by $g(x) = xb$. Then $fg(x) = xba = xp = x$ for every $x \in M$, therefore $fg = 1$ (the identity map of M). We have proved that in the semigroup S of all G-self-maps of M, every element has a right inverse. Hence S is a group. (Alternatively, we first deduce from the equality $fg = 1$ that all elements of S are surjective and then, applying this to g, we see that f is also injective.) □

We are now in a position to prove that *every universal compact minimal G-space is isomorphic to M*. First note that the minimal compact G-space M is itself universal: if Y is any compact G-space, there exists a G-map of the greatest ambit X to Y, and its restriction to M is a G-map of M to Y. Now let M' be another universal compact minimal G-space. There exist G-maps $f : M \to M'$ and $g : M' \to M$. Since M' is minimal, f is surjective. On the other hand, in virtue of Proposition 3.3 the composition $gf : M \to M$ is bijective. It follows that f is injective and hence a G-isomorphism between M and M'.

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